Center for Turbulence Research
Proceedings of the Summer Program 1987

N88-23102

On Local Approximations of the Pressure-Strain Term in Turbulence Models

By P. BRADSHAW[1], N. N. MANSOUR[2] and U. PIOMELLI[3]

The reason for the success of the approximation that uses the mean velocity gradient $\partial U/\partial y$ outside the integral solution of the Poisson equation for pressure, and sets the gradient equal to its value at the point where the pressure is being calculated, has been explored. This approximation is implicit in most existing turbulence models, where the pressure-strain terms are assumed to be functions of local variables rather than of the proper spatial integrals. Direct simulation data for the channel were used to evaluate spatial correlations of pressure and velocity gradients. The results show that a correlation coefficient of about -0.5 between the rapid pressure and its Laplacian (proportional to $\partial v'/\partial x$); this is in favor of the local assumption. Analysis of the solution to the Poisson equation indicates that the assumption will be valid when the mean velocity gradient varies slowly as compared to the correlation length of the fluctuating velocity gradients.

1. Introduction

The pressure-strain "redistribution" terms in the Reynolds-stress transport equations, which are mean products of the pressure fluctuation and various components of the fluctuating rate of strain, have received much attention from turbulence modelers for two reasons: first, the pressure-strain terms in the shear stress equations are the largest of the terms that need to be modelled in those all-important equations; secondly, little experimental data is available to properly assess the models.

In an experimental setup, pressure fluctuations within a boundary layer cannot be measured with any assurance of accuracy, because the velocity fluctuations induce pressure fluctuations on any solid probe inserted in the flow, and these spurious fluctuations are usually larger than those in the undisturbed flow. Using the instantaneous velocity field in the entire domain, turbulence simulations (numerical solutions of the full time-dependent Navier-Stokes equations without any modeling approximations) can be used to deduce statistics of measurable and of "unmeasurable" quantities such as those we will discuss in the next section. We shall discuss in this paper an approximation to the pressure-strain term which is commonly used in modeling the Reynolds-stress transport equations.

1 Imperial College
2 NASA Ames Research Center
3 Stanford University

PRECEDING PAGE BLANK NOT FILMED

2. Local approximation to the pressure

For an incompressible flow the fluctuating pressure can be obtained from the velocity field by solving the Poisson equation

$$\frac{\partial^2 p}{\partial x_j \partial x_j} = -2\frac{\partial U_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} - \left(\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} - \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i}}\right) \tag{1a}$$

with the boundary condition at the walls

$$\frac{\partial p}{\partial y} = \frac{1}{Re}\frac{\partial^2 v'}{\partial y^2}. \tag{1b}$$

Equation (1a) shows that any correlations of the pressure with other fluctuating quantities will have a part proportional to the mean velocity gradients $\partial U_i/\partial x_j$ and a part depending only on velocity fluctuations. These are called the "rapid" and "slow" parts respectively, because only the former will respond at once to a change in the mean velocity field. An inhomogeneous boundary condition makes the split less obvious. However, for the case of the fully developed channel the volume integrals of each of the terms on the right hand side of (1a) will integrate to zero and pressure in this case can be split in three parts: the rapid pressure p^1, which satisfies

$$\frac{\partial^2 p^1}{\partial x_j \partial x_j} = -2\frac{\partial U_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} \tag{2a}$$

with the boundary condition at the walls

$$\frac{\partial p^1}{\partial y} = 0; \tag{2b}$$

the slow pressure p^2, which is the solution to:

$$\frac{\partial^2 p^2}{\partial x_j \partial x_j} = -\left(\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} - \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i}}\right) \tag{3a}$$

with the boundary condition at the walls

$$\frac{\partial p^1}{\partial y} = 0; \tag{3b}$$

and, finally, the Stokes pressure p^S, which is the solution to Laplace's equation with the boundary condition (1b) at the walls.

The pressure-strain terms that appear in the Reynolds stress equations are linear in p, so that the total term will be equal to the sum of the rapid pressure-strain, slow

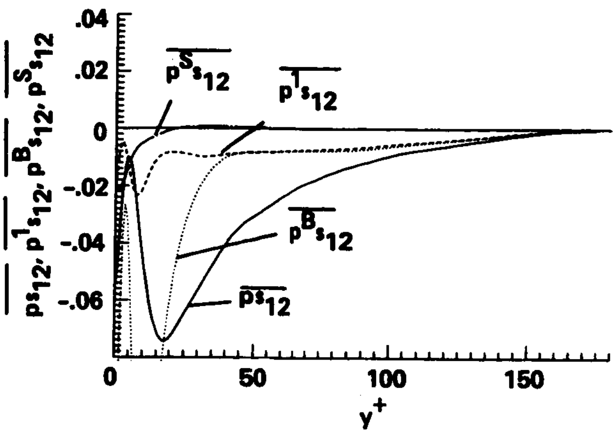

FIGURE 1. Pressure-strain term in the $-\overline{uv}$ transport equation.

pressure-strain and Stokes pressure-strain. The solution of (2a) with the boundary condition (2b) at the walls is

$$p^1 = -\frac{1}{4\pi}\int_V 2\frac{\partial U_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} G dV \tag{4}$$

in which G is Green's function with homogeneous Neumann boundary conditions at the walls. Note that most modelers neglect the surface integral terms that should be added to (4) if non-homogeneous Neumann conditions are used for the pressure. The use of the homogeneous boundary condition (2b) at the walls for the rapid pressure is consistent with (4) and the approximation used by the modelers. Although, by (4), the rapid pressure is an integral of the weighted mean velocity gradients over the entire domain, in modeling the assumption is often made that p^1 depends only on the local mean velocity gradient, which allows one to take the mean velocity gradients outside of the integral in (4) to yield the following approximation for the pressure:

$$p^B = -\frac{1}{4\pi}2\frac{\partial U_i}{\partial x_j}\int_V \frac{\partial u'_j}{\partial x_i} G dV. \tag{5}$$

Direct experimental measurement of either p^1 or p^B is impossible, but the use of direct simulation results allows us to test the approximation (5) directly by computing the integrals in (4) and (5) and comparing the approximate value p^B with p^1. The purpose of the present paper is to examine the results of this comparison for channel flow at low Reynolds number.

3. Results

The results of the direct numerical simulation of fully developed channel flow at $Re_\tau = 180$ (Kim, Moin and Moser, 1987) were used to compute all the statistics presented in this work.

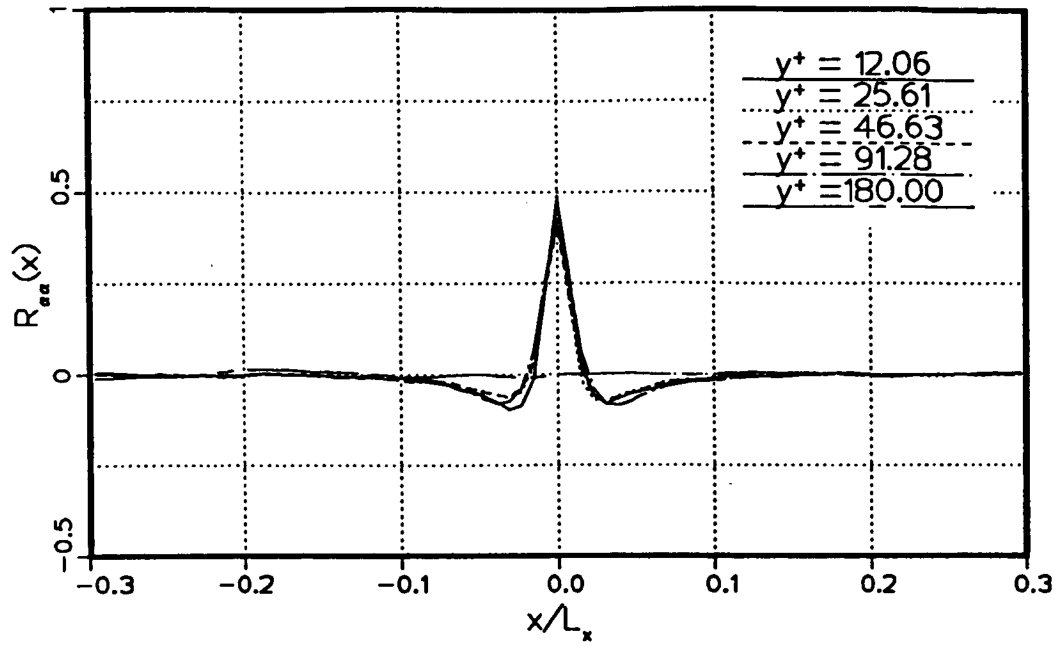

FIGURE 2. Two point correlation of the rapid-pressure term with $\partial v/\partial x$.

Figure 1 shows the contribution of the rapid pressure-strain term in the $-\overline{u'v'}$ transport equation. All results are normalized by $\rho u_\tau^4/\nu$: on this scale the maximum rate of production of turbulent energy, which occurs where $\tau = \tau_w/2$, *i.e.* at about $y^+ = 12$, is 0.25. Note that the plotted terms are negative if $-\overline{u'v'}$ is being reduced and positive if it is being increased by the pressure-strain redistribution.

The accuracy of the local approximation to the pressure-strain terms is surprisingly good: differences between the exact and approximate values are generally within the scatter of the statistical averaging, except in the viscous sublayer ($y^+ < 30$). The rapid pressure-strain contribution to the other Reynolds stress transport equations shows similar trends.

To further investigate the reasons for the success of the approximation outside of the sublayer, as well as to ascertain the causes of its failure inside it, we examined the correlation coefficient between the rapid pressure p^1 and various components of the fluctuating strain-rate tensor for varying distances from the wall.

The most noticeable feature of the simulation results is the high correlation between pressure and velocity gradients. In particular, the coefficient of correlation between p^1 and $\partial v'/\partial x$ (Figure 2) is numerically as large as 0.5. This implies a correlation coefficient of -0.5 between the pressure and its Laplacian. The relatively high value for p^1 is noteworthy. It is not high enough to legitimize the p^B

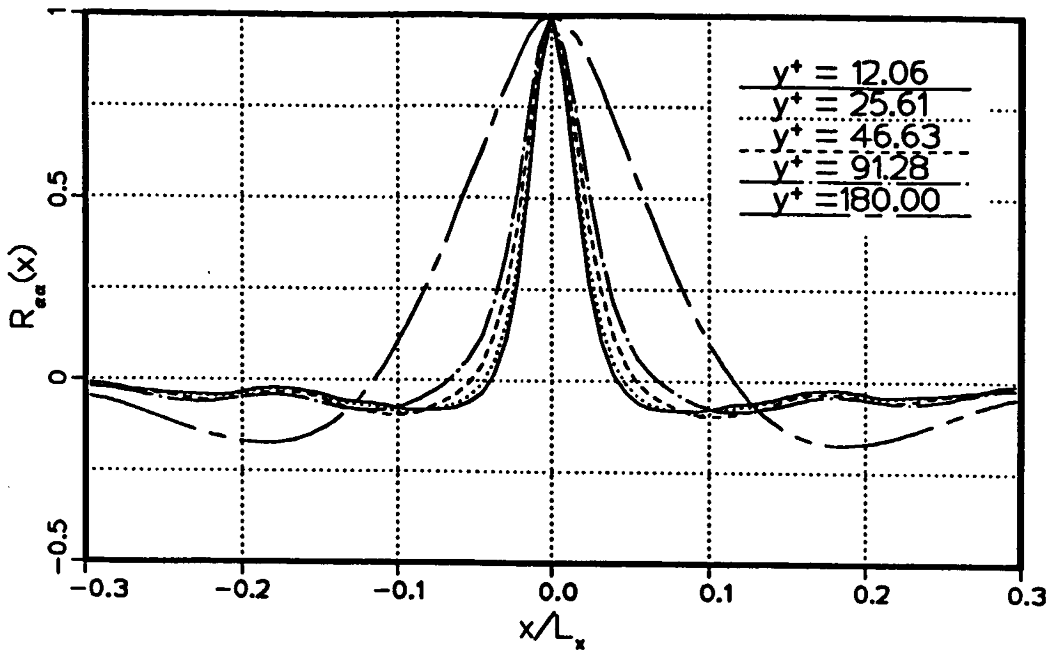

FIGURE 3. Two point auto-correlation of the rapid-pressure.

approximation directly, but it suggests that the rapid pressure is not so dependent on contributions from distant parts of the flow as its governing Poisson equation nominally implies. A demonstration of this dependency is the rapid increase of the length scale of p^1 near the channel centerline, which can be inferred from the auto-correlation of p^1, Figure 3. Near the centerline the source term in the Poisson equation for p^1, being proportional to the mean shear, vanishes; contributions to p^1 on the centerline, therefore, must necessarily come from regions off the centerline, and have their high wavenumber parts attenuated by distance.

For homogeneous flows in the x and z directions, the pressure-strain term can be written as follows:

$$\overline{p^1 \frac{\partial u_i'}{\partial x_j}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p^1(x,y,z)\frac{\partial u_i'}{\partial x_j}(x,y,z)dxdz. \tag{6}$$

Substitution of (4) into (6) gives

$$\overline{p^1 \frac{\partial u_i'}{\partial x_j}} = \int_{-1}^{1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial U}{\partial y}(y')\times$$
$$\frac{\partial v'}{\partial x}(x-x',y',z-z')\frac{\partial u_i'}{\partial x_j}(x,y,z)G(x',y,y',z')dxdzdx'dz'dy' \tag{7}$$

whereas the local approximation (5) yields

$$\overline{p^B \frac{\partial u_i}{\partial x_j}} = \frac{\partial U}{\partial y}(y) \int_{-1}^{1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial v'}{\partial x}(x - x', y', z - z') \frac{\partial u_i'}{\partial x_j}(x, y, z) G(x', y, y', z') dx' dz' dx dz dy'. \quad (8)$$

The two inner integrals in (7) and (8) represent the two-point correlation between $\partial v(y')/\partial x$ and $\partial u_j(y)/\partial x_i$, weighted by Green's function G, as a function of y and y'. This correlation can be expected to vanish as $y - y'$ increases, since Green's function decreases as $y - y'$ increases and the velocity gradients are not correlated over large distances. As long as the length scale of this correlation is small with respect to the length over which the mean shear can be considered constant, the local approximation will hold. In the logarithmic region the mean shear varies slowly ($\partial^2 U/\partial y^2 \sim 1/y^2$) and the local approximation holds. At the sublayer edge, where the velocity gradient changes significantly, this is not true and the local approximation fails.

4. Conclusions

The results of numerical simulations of turbulent channel flows have been used to examine the validity of the local approximation of the pressure-strain term in the Reynolds stress transport equation.

Outside of the viscous sublayer the local approximation compares very well with the exact pressure strain. This agreement is due, at least in part, to the high correlation between the rapid pressure and its Laplacian, which suggests that only the nearer parts of the flow contribute to the rapid pressure at a point.

In the viscous sublayer the distance over which the mean shear can be considered constant is comparable to the length scale in the normal direction of the correlations of velocity gradients, leading to failure of the local approximation.

REFERENCES

Kim, J., Moin, P. & Moser, R. D. 1987 Turbulence statistics in fully developed channel flow at low Reynolds number. *J. Fluid Mech.* **177**, 133–166.